Decoupling of DeGiorgi-type systems via multi-marginal optimal transport*

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Abstract

We exhibit a surprising relationship between elliptic gradient systems of PDEs, multi-marginal Monge-Kantorovich optimal transport problem, and multivariable Hardy-Littlewood inequalities. We show that the notion of an *orientable* elliptic system, conjectured in [6] to imply that (in low dimensions) solutions with certain monotonicity properties are essentially 1-dimensional, is equivalent to the definition of a *compatible* cost function, known to imply uniqueness and structural results for optimal measures to certain Monge-Kantorovich problems [11]. Orientable nonlinearities and compatible cost functions turned out to be also related to submodular functions, which appear in rearrangement inequalities of Hardy-Littlewood type. We use this equivalence to establish a decoupling result for certain solutions to elliptic PDEs and show that under the orientability condition, the decoupling has additional properties, due to the connection to optimal transport.

1 Introduction

The main purpose of this note is to pinpoint a surprising connection between elliptic systems of PDEs, multi-marginal optimal transportation, and multivariable extended Hardy-Littlewood inequalities. A recent paper by Fazly and Ghoussoub [6] introduced the concept of an *orientable* elliptic system (Definition 2.1 below), which seems to be the appropriate framework for investigating De Giorgi type conjectures ([7], [2]) for systems of more than two equations. On the other hand, the thesis of the second author [11], following work of Carlier [5], introduced the concept of a *compatible* cost function (Definition 2.2 below), a natural, covariant condition ensuring uniqueness and structural results on solutions to a multi-marginal optimal transportation problem with one dimensional marginals. These notions turned out to be also related to submodular (or 2-monotone) functions, which appear in rearrangement inequalities of Hardy- Littlewood type as studied by several authors dating back to Lorentz [9].

We will show here that these three conditions are actually equivalent. As a consequence we shall see how the concept of an H-monotone solution to the system (1) below, which was introduced in [6], is intimately related to the geometric structure of optimal measures in the optimal transport problem (2) uncovered in [11]. We also show that monotone solutions to the elliptic system can be *decoupled*. If the solution is H-monotone, we use the connection with optimal transportation to show that the sum of the decoupled non-linearities is everywhere less than the original non-linearity.

Let $H : \mathbb{R}^m \to \mathbb{R}$ be a C^2 function. We will consider the system of elliptic PDEs on $\mathbb{R}^N$:

$$\Delta u = \nabla H(u), \tag{1}$$

where $u = (u_1, u_2, ..., u_m)$ represents an m-tuple of functions. In this context, H is often referred to as the *non-linearity* of the system.

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Given probability measures $\mu_1, \mu_2, ..., \mu_m$ on $\mathbb{R}$, the optimal transport (or Monge-Kantorovich) problem consists of minimizing

$$\int_{\mathbb{R}^m} H(p_1, p_2, ..., p_m) d\gamma(p_1, p_2, ..., p_m) \tag{2}$$

among probability measures γ on $\mathbb{R}^m$ whose 1-dimensional marginals are μ_i. In this setting, H is called the *cost function*.

If H is bounded below on $\mathbb{R}^m$, then there exists a solution $\bar{\gamma}$ to the Kantorovich problem (2), as well as an m-tuple of functions $(V_1, V_2, ..., V_m)$ – called Kantorovich potentials – such that for all $i = 1, ..., m$,

$$V_i(p_i) = \inf_{\substack{p_j \in \mathbb{R} \\ j \neq i}} \Big(H(p_1, p_2, ..., p_m) - \sum_{j \neq i} V_j(p_j) \Big), \tag{3}$$

and which maximizes the following dual problem

$$\sum_{i=1}^{m} \int_{\mathbb{R}} V_i(p_i) d\mu_i \tag{4}$$

among all m-tuples $(V_1, V_2, ..., V_m)$ of functions $V_i \in L^1(\mu_i)$ for which

$$\textstyle\sum_{i=1}^m V_i(p_i) \leq H(p_1, ..., p_m) \text{ for all } (p_1, ..., p_m) \in \mathbb{R}^m. \tag{5}$$

Furthermore, the maximum value in (4) coincides with the minimum value in (2) and

$$\sum_{i=0}^{m-1} V_i(p_i) = H(p_1, ..., p_m) \quad \text{ for all } (p_1, ..., p_m) \in \text{support}(\bar{\gamma}). \tag{6}$$

In a certain sense, the dual problem (4) provides a decoupling of the original Monge-Kantorovich problem. In this note, we shall show that the above scheme also provides a decoupling of the system (1), at least for certain type of solutions. Roughly speaking, under certain monotonicity conditions on a solution $u = (u_1, u_2, ..., u_m)$ of (1), the duality in the Monge-Kantorovich problem applied to a suitable set of marginals $(\mu_{u_1}, \mu_{u_2}, ..., \mu_{u_2})$ associated to u, leads to a decoupled system

$$\Delta u_i = \tfrac{\partial V_i}{\partial p_i}(u_i(x)) \ \text{ for } i = 1, ..., m. \tag{7}$$

having $u = (u_1, u_2, ..., u_m)$ as a solution, where $V_1, ..., V_m$ are the corresponding Kantorovich potentials. Decoupled systems are much simpler than coupled systems and have a number of advantages. For example, whenever the decoupling above is possible and the potentials V_i are positive, one can deduce the following Modica inequality [10] for systems

$$\textstyle\sum_{i=1}^m |\nabla u_i(x)|^2 \leq 2H(u_1(x), ..., u_m(x)) \text{ for all } x \in \mathbb{R}^N. \tag{8}$$

Whether such an inequality holds true for a general gradient system remains an open problem. See Caffarelli-Lin [4] and Alikakos [1].

The compatibility condition is also equivalent –up to a change of variables– to the classical notion of submodularity (also known as 2-monotonicity) of H (see Definition 2.3). A result of Carlier on the structure of optimizers in (2) for a submodular cost function H is essentially equivalent to a rearrangement inequality of Hardy-Littlewood type [5][9][3]. We shall see that the covariant analogue of this result noted in [11] implies a rearrangement inequality for compatible H, but where decreasing rearrangements should be replaced by *H-monotone rearrangements*. Note that rearrangement inequalities have found applications in nonlinear optics [8], where one looks for ground states of energy functionals of the form

$$E(u) = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i=1}^{m} |\nabla u_i(x)|^2 + H(u) dx. \tag{9}$$

Note that (1) is nothing but the Euler-Lagrange system corresponding to this energy functional.

For a submodular H, any nonnegative function u may be rearranged in a symmetric and decreasing way (that is, the decreasing rearrangement is with respect to the variable $|x|$) to decrease the total energy. In our case, where H is orientable, any u can be rearranged in an H-monotone way with respect to the variable x_N to decrease the second component of the energy. In $N = 1$ dimensions, this rearrangement will also decrease the gradient term, so that the total energy also decreases under H-monotone rearrangement.

Finally, we note that, while these types of decouplings can only be done for gradient systems, it is not essential to have the Laplacian on the left hand side; similar decouplings are possible when the left hand side is replaced by any decoupled differential operator $D_i u_i$.

2 Orientable systems, compatible cost and sub-modular functions

We begin by recalling the definitions of an orientable system and of a compatible cost. Our definitions here are actually strict versions of the original definitions in [6], where non-strict inequalities are used.

Definition 2.1. *The system (or the non-linearity H) is called orientable if there exist constants θ_i such that, for all $i \neq j$,*

$$\theta_i \theta_j \frac{\partial^2 H}{\partial p_i \partial p_j} < 0.$$

Next, we recall the definition of compatibility, discussed in [11].

Definition 2.2. *We say H is compatible if for all distinct i, j, k, we have*

$$\tfrac{\partial^2 H}{\partial p_i \partial p_j} \left(\tfrac{\partial^2 H}{\partial p_k \partial p_j} \right)^{-1} \tfrac{\partial^2 H}{\partial p_k \partial p_i} < 0 \text{ for all } p = (p_1, p_2, ..., p_m) \ .$$

Next, recall the following classical definition.

Definition 2.3. *The function H is submodular (or 2-increasing in Economics) if*

$$H(\mathbf{x} + h\mathbf{e}_i + k\mathbf{e}_j) + H(\mathbf{x}) \leq H(\mathbf{x} + h\mathbf{e}_i) + H(\mathbf{x} + k\mathbf{e}_j) \qquad (i \neq j, \quad h, k > 0),$$

where $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^m$.

Lemma 2.4. *The following are equivalent for a function $H \in C^2(\mathbb{R}^m, \mathbb{R})$.*

1. *H is orientable.*
2. *H is compatible.*
3. *After a change of variables, H is submodular.*

Proof. We will first show that 1) and 2) are equivalent. First suppose H is orientable. Then we have, for all distinct i, j, k,

$$\frac{\partial^2 H}{\partial p_i \partial p_j} \theta_i \theta_j \left(\frac{\partial^2 H}{\partial p_k \partial p_j} \theta_k \theta_j \right)^{-1} \frac{\partial^2 H}{\partial p_k \partial p_i} \theta_k \theta_i < 0,$$

and therefore

$$\theta_i^2 \frac{\partial^2 H}{\partial p_i \partial p_j} \left(\frac{\partial^2 H}{\partial p_k \partial p_j} \right)^{-1} \frac{\partial^2 H}{\partial p_k \partial p_i} < 0,$$

which implies compatibility.

On the other hand, if H is compatible, set $\theta_i = 1$ if $\frac{\partial^2 H}{\partial p_i \partial p_1} > 0$ and $\theta_i = -1$ if $\frac{\partial^2 H}{\partial p_i \partial p_1} < 0$. Then, up to positive multiplicative constants, we have

$$\theta_i \theta_j \frac{\partial^2 H}{\partial p_i \partial p_j} = \frac{\partial^2 H}{\partial p_i \partial p_1} \frac{\partial^2 H}{\partial p_1 \partial p_j} \frac{\partial^2 H}{\partial p_i \partial p_j} < 0,$$

establishing H-orientability.

Now we will prove the equivalence of 2) and 3). First, we recall that a smooth function H is submodular if

$$\frac{\partial^2 H}{\partial p_i \partial p_j} < 0 \text{ for all } i \neq j \text{ and } p = (p_1, p_2, ..., p_m). \tag{10}$$

As was noted in [11], compatibility is equivalent to the existence of changes of variables $p_i \mapsto q_i$ such that $H(q_1, q_2, ..., q_m)$ satisfies (10). It is therefore invariant under this sort of transformation. Assuming now that H is compatible, define a change of coordinates as follows: set $q_1 = p_1$ and, for $i \geq 2$, set $q_i = p_i$ if $\frac{\partial^2 H}{\partial p_1 \partial p_i} < 0$ and $q_i = -p_i$ if $\frac{\partial^2 H}{\partial p_1 \partial p_i} > 0$. It is then clear that $\frac{\partial^2 H}{\partial q_1 \partial q_i} < 0$ for all i. For any distinct $i, j \neq 1$, we then have by the compatibility condition,

$$\frac{\partial^2 H}{\partial q_i \partial q_j}\Big(\frac{\partial^2 H}{\partial q_1 \partial q_j}\Big)^{-1}\frac{\partial^2 H}{\partial q_1 \partial q_i} < 0,$$

which easily yields $\frac{\partial^2 H}{\partial q_i \partial q_j} < 0$.

On the other hand, to see that 3) implies 2), it is sufficient to note that submodularity implies compatibility, and that compatibility is invariant under changes of coordinates of the form $p_i \mapsto q_i(p_i)$. $\square$

We note that the condition of submodularity on H is essentially equivalent to the following extended Hardy-Littlewood inequality: for all choices of real-valued non-negative measurable functions $(u_1, ..., u_m)$ that vanish at infinity, we have

$$\int_{\mathbb{R}^N} H(u_1^*(x), ..., u_m^*(x))dx \leq \int_{\mathbb{R}^N} H(u_1(x), ..., u_m(x)dx, \tag{11}$$

where u_i^* is the symmetric decreasing rearrangement of u_i for $i = 1, ..., m$. See for example Burchard-Hajaiej [3] and the references therein.

We now describe the properties of multi-marginal mass transport under a compatible function cost H. Given probability measures $\mu_1, \mu_2, ..., \mu_m$ on $\mathbb{R}$, and if $\frac{\partial^2 H}{\partial q_i \partial q_j} < 0$ (i.e., H is submodular relative to the q variables), then as shown by Carlier [5], there is a unique solution to the optimal transportation problem (2), given by $\gamma = (I, f_2, f_3, ... f_m)_\# \mu_1$, where $f_i : \mathbb{R} \to \mathbb{R}$ is the unique increasing map pushing forward μ_1 to μ_i. If now H is compatible in the original p coordinates, then the unique solution to (2) is $\gamma = (Id, g_1, g_2, ..., g_m)_\# \mu_1$, where $g_i : \mathbb{R} \to \mathbb{R}$ is *increasing* if $\frac{\partial^2 H}{\partial p_1 \partial p_i} < 0$ and *decreasing* if $\frac{\partial^2 H}{\partial p_1 \partial p_i} > 0$. It is also the unique such map pushing forward μ_1 to $(\mu_1, \mu_2, ..., \mu_m)$. For proofs and more discussion of the compatibility condition, see [11].

We now discuss the notion of H-monotonicity introduced by Fazly-Ghoussoub in [6].

Definition 2.5. *1. A function $u = (u_1, u_2, ..., u_m) \in C^1(\mathbb{R}^N; \mathbb{R}^m)$ is said to be monotone if each u_i is strictly monotone with respect to x_N; that is, if $\frac{\partial u_i}{\partial x_N} \neq 0$.*

2. u is said to be H-monotone if it is monotone and if for all $i \neq j$,

$$\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial u_i}{\partial x_N}\frac{\partial u_j}{\partial x_N} < 0.$$

It is easy to see that the existence of an H-monotone function implies that H is necessarily orientable. Conversely, we shall see below that if H is orientable, then one can associate to any function $u = (u_1, u_2, ..., u_m)$, its H-monotone rearrangement. But first, let's note the following easy application of the above lemma coupled with Carlier's result. For our purposes it will often be useful to decompose $x \in \mathbb{R}^N$ into $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$.

Proposition 2.6. *Let $u = (u_1, u_2, ..., u_m)$ be a bounded monotone function in $C^1(\mathbb{R}^N; \mathbb{R}^m)$. Let μ be a probability measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure. For each $x' \in \mathbb{R}^{N-1}$, let $\mu_i^{x'}$ be the pushforward of μ by the map $u_i^{x'} : x_N \mapsto u_i(x', x_N)$ and set $\gamma^{x'} := (u_1^{x'}, u_2^{x'}, ..., u_m^{x'})_\# \mu$. Then the following are equivalent:*

1. *u is H-monotone.*
2. *For each $x' \in \mathbb{R}^{N-1}$, the measure $\gamma^{x'}$ is optimal for the Monge-Kantorovich problem (2), when the marginals are given by $\mu_i^{x'}$.*

Proof. 1) $\to$ 2) is obvious. Assuming now 2), we note that for each $x' \in \mathbb{R}^{N-1}$, the measures $\mu_i^{x'}$ are absolutely continuous with respect to Lebesgue measure by the (strict) monotonicity of the u_i. The support of the optimizer $\gamma^{x'}$ must be H-monotone, by the equivalence of H orientability and H compatibility, combined with the result in [11]. As this support is exactly the image of $(u_1^{x'}, u_2^{x'}, ..., u_m^{x'})$, this implies

$$\frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial u_i}{\partial x_N} \frac{\partial u_j}{\partial x_N} < 0.$$

As this holds for any x', this completes the proof. □

The above proposition leads to the following notion:

Definition 2.7. For a function $u = (u_1, u_2, ..., u_m)$ in $C(\mathbb{R}^N; \mathbb{R}^m)$, we define its *$H$-monotone rearrangement* $u^*(x', x_N) = (u_1^*, u_2^*, ..., u_m^*)$ *with respect to* x' as follows: For each fixed $x' \in \mathbb{R}^{N-1}$, denote by $\mu_i^{x'}$ the push-forward of Lebesgue measure μ on $\mathbb{R}$ by $x_N \mapsto u_i(x' x_N)$, and define

- For $i = 1$, $x_N \mapsto u_1^*(x', x_N)$ to be the unique monotone decreasing map which maps μ to $\mu_1^{x'}$.
- For $i \neq 1$, then
  - If $\frac{\partial^2 H}{\partial p_i \partial p_1} < 0$, let $x_N \mapsto u_i^*(x', x_N)$ be the unique monotone increasing map pushing μ to $\mu_i^{x'}$.
  - If $\frac{\partial^2 H}{\partial p_i \partial p_1} > 0$, let $x_N \mapsto u_i^*(x', x_N)$ be the unique monotone decreasing map pushing μ to $\mu_i^{x'}$.

By combining the above Lemma with a change of variables and the rearrangement inequality (11) for submodular functions, we obtain that if H is orientable, then

$$\int_{\mathbb{R}} H(u^*(x', x_N)) dx_N \leq \int_{\mathbb{R}} H(u(x', x_N)) dx_N.$$

Integrating over x' then yields

$$\int_{\mathbb{R}^N} H(u^*(x', x_N)) dx \leq \int_{\mathbb{R}^N} H(u(x', x_N)) dx.$$

That is, the H-monotone rearrangement lowers the second term in the energy functional $E(u)$ in equation (9). As to the gradient term in $E(u)$, it is known that it decreases under monotone symmetric rearrangements. On the other hand, it does not necessarily under H-monotone rearrangements, unless $N = 1$. We therefore have the following.

Proposition 2.8. *If H is orientable, then for any $u = (u_1, u_2, ..., u_m)$, we have*

1. *$\int_{\mathbb{R}^N} H(u^*(x))dx \leq \int_{\mathbb{R}^N} H(u(x)dx$, where u^* is the H-monotone rearrangement of u.*
2. *If u is one-dimensional, i.e. $u \in C^1(\mathbb{R}; \mathbb{R}^m)$, then $E(u^*) \leq E(u)$, where*

$$E(u) = \int_{\mathbb{R}} \left(\frac{1}{2} \sum_{i=1}^{m} |\frac{du_i}{dx}|^2 + H(u) \right) dx,$$

and consequently, the system of ODEs,

$$u_{xx} = \nabla H(u) \text{ on } \mathbb{R}, \tag{12}$$

has an H-monotone solution, whenever E attains its infimum on $H^1(\mathbb{R}; \mathbb{R}^m)$.

3 Decoupling systems in the presence of H-monotone solutions

Fazly and Ghoussoub [6] conjectured, and proved in dimensions $N \leq 3$, that H-monotone solutions of the system of elliptic PDEs on $\mathbb{R}^N$,

$$\Delta u = \nabla H(u), \tag{13}$$

where $u = (u_1, u_2, ..., u_m)$ represents an m-tuple of functions, must be essentially 1-dimensional; that is, each u_i takes the form $u_i(x) = g_i(a_i \cdot x' - x_N)$ for some $a_i \in \mathbb{R}^{N-1}$. This is a systems analogue of DeGiorgi's famous conjecture for monotone solutions of the Allen-Cahn equation [7] [2].

They also showed that, for orientable systems in dimension $N = 2$, all the components of a *stable solution* $u = (u_1, u_2, ..., u_m)$ have common level sets, which also happened to be hyperplanes. We shall therefore say that the components $(u_i)_{i=1}^m$ have common level sets if for any $i \neq j$ and any $\lambda \in \mathbb{R}$, there exists some $\bar{\lambda}$ such that

$$\{x : u_i(x) = \lambda\} = \{x : u_j(x) = \bar{\lambda}\}.$$

One might therefore conjecture that stable solutions of orientable systems may always have common level sets in higher dimension N, if not necessarily hyperplanes. This can be seen as complementary to the conjecture of Fazly and Ghoussoub, which asserts that –at least in low dimensions (say $N \leq 8$) – the level sets of H-monotone solutions are hyperplanes (but possibly different hyperplanes for each component u_i of the solution).

We now establish a result on the decoupling of the system (13).

Theorem 3.1. *Let $u = (u_1, ..., u_m)$ be a bounded solution to system* (13).

1. *If $u = (u_1, ..., u_m)$ is monotone, then there exist functions $V_i(p_i, x')$ such that u solves the system of decoupled equations:*
$$\Delta u_i = \tfrac{\partial V_i}{\partial p_i}(u_i(x), x') \quad \text{for } i = 1, ..., m. \tag{14}$$
Furthermore, along the solution, we have
$$\textstyle\sum_{i=1}^m V_i(u_i(x), x') = H(u_1(x), u_2(x), ..., u_m(x)) \text{ for } x \in \mathbb{R}^N. \tag{15}$$

2. *If $u = (u_1, ..., u_m)$ is H-monotone, then for all $p = (p_1, p_2, ..., p_m) \in \mathbb{R}^m$,*
$$\sum_{i=1}^m V_i(p_i, x') \leq H(p_1, p_2, ..., p_m). \tag{16}$$

3. *If the u_i have common level sets, then the V_i can be chosen to be independent of x', that is $V_i(p_i, x') = V_i(p_i)$.*

Proof. Fix $x' \in \mathbb{R}^{N-1}$, and define $V_i(\cdot, x')$ on the range of $x_N \mapsto u_i(x', x_N)$ as follows. Choose the unique x_N such that $p_i = u_i(x', x_N)$ and set

$$\frac{\partial V_i}{\partial p_i}(p_i, x') = \frac{\partial H}{\partial p_i}(u(x', x_N)).$$

It then follows by construction that

$$\Delta u_i = \frac{\partial H}{\partial p_i}(u(x', x_N)) = \frac{\partial V_i}{\partial p_i}(u_i, x').$$

Note that, as each $V_i(x', \cdot)$ is defined only up to an arbitrary constant, we can choose the constants so that for $x_N = 0$,

$$\sum_{i=1}^m V_i(u_i(x', 0), x') - H(u_1(x', 0), u_2(x', 0), ..., u_m(x', 0)) = 0.$$

Now note that by the definition of V_i, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_N}\Big(\sum_{i=1}^m V_i(u_i(x',x_N),x') - H(u_1(x',x_N),u_2(x',x_N),...,u_m(x',x_N))\Big)\\
= &\sum_{i=1}^m \frac{\partial V_i}{\partial p_i}(u_i(x',x_N),x')\frac{\partial u_i}{\partial x_N} - \sum_{i=1}^m \frac{\partial H}{\partial p_i}(u_1(x',x_N),u_2(x',x_N),...,u_m(x',x_N))\frac{\partial u_i}{\partial x_N}\\
= &\ 0.
\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^m V_i(u_i(x',x_N),x') - H(u_1(x',x_N),u_2(x',x_N),...,u_m(x',x_N)) = 0$$

everywhere on $\mathbb{R}^N$.

2) Now suppose that u is H-monotone. For fixed $x' \in \mathbb{R}^{N-1}$, let $\gamma^{x'}$ and $\{\mu_i^{x'}, i=1,...,m\}$ be defined as in Proposition 2.6. The measure $\gamma^{x'}$ is then optimal for the Monge-Kantorovich problem (2) with given marginals $\mu_i^{x'}, i=1,...,m$. In this case, the V_i defined above play the role of Kantorovich potentials, and so we have

$$\sum_{i=1}^m V_i(p_i,x') \le H(p_1,p_2,...,p_m).$$

everywhere.

3) In this case, the image of the map $(u_1^{x'},u_2^{x'},...,u_m^{x'})$ is independent of x' . The measures $\gamma^{x'}$ are then all supported on the same set, and so we can choose the $V_i(p_i,x') = V_i(p_i)$ to be independent of x'. The image of $(u_1,u_2,...,u_m)$ is equal to the image of $(u_1^{x'},u_2^{x'},...,u_m^{x'})$ for any fixed x', and the results in [11] imply that any measure supported on this set is optimal for its marginals. □

Remark 3.2. If u is H-monotone such that all its components $(u_i)_i$ have common level sets, then the image of $(u_1,u_2,...,u_m)$ is a 1-dimensional set, and, by H-monotonicity, the results in [11] imply that any measure supported on this set is optimal for its marginals. In particular, for any measure μ on $\mathbb{R}^N$, absolutely continuous with respect to Lebesgue measure, the measure $\gamma = (u_1,u_2,...,u_m)_\#\mu$ is optimal for its marginals $\mu_i = u_{i\#}\mu$.

As an immediate application of Theorem 3.1, we obtain the following Modica type estimate.

Corollary 3.3. *Suppose that u is a bounded monotone solution of (13) such that all its components $(u_i)_i$ have common level sets. Then, letting $\bar{V}_i = \min_{p_i\in Range(u_i)} V_i(p_i)$, we have*

$$\frac{1}{2}\sum_{i=1}^m |\nabla u_i(x_i)|^2 \le H(u) - \sum_{i=1}^m \bar{V}_i. \tag{17}$$

Proof. By Theorem 3.1, u solves the decoupled system

$$\Delta u_i = \frac{\partial V_i}{\partial p_i}(u_i).$$

Then, by Modica's well known inequality, we have $\frac{1}{2}|\nabla u_i(x_i)|^2 \le V_i(u_i) - \bar{V}_i$. Summing over i and using Theorem 3.1, part 1), completes the proof. □

Remark 3.4. This corollary applies to, for example, bounded, monotone solutions in $\mathbb{R}^2$ with an orientable H. Fazly and Ghoussoub [6] showed that in this case bounded stable [1] solutions satisfy

$$\nabla u_j = C_{i,j}\nabla u_i \tag{18}$$

[1] See [6] for the definition of stability.

for constants $C_{i,j}$ with signs opposite the signs of $\frac{\partial^2 H}{\partial p_i \partial p_j}$ [6]. When the level sets of $u_i(x', x_N) = g_i(a_i \cdot x' - x_N)$ are hyperplanes, this implies that $a_i = a_j := a$ for all i, j. Therefore, the u_i's share the same level sets and the corollary applies.

In fact, in this case we can prove even more; we can remove the $\bar{V}_i$ terms from the inequality. A simple computation yields

$$(|a|^2 + 1)g_i''(a \cdot x' - x_N) = \Delta u_i(x) = V_i'(g_i(a \cdot x' - x_N)).$$

Denoting the argument $a \cdot x' - x_N$ by y, multiplying the equation by $g'(y)$ and integrating, we obtain

$$\frac{1}{2}(|a|^2 + 1)g_i'(y)^2 = V_i(g_i(y)) - C_i, \tag{19}$$

for some constants C_i. As the solution $g_i(a \cdot x' - x_N)$ is bounded and monotone, we can take the limit as $y \to \infty$; as we must have $\lim_{y\to\infty} g'(y) = 0$, we obtain $C_i = V_i(\lim_{y\to\infty} g_i(y))$. Now, summing over i in (19) yields

$$\frac{1}{2}(|a|^2 + 1)\sum_{i=1}^{m} g_i'(y)^2 = \sum_{i=1}^{m} V_i(g_i(y)) - \sum_{i=1}^{m} C_i = H(g_1(y), g_2(y), ..., g_m(y)) - \sum_{i=1}^{m} C_i. \tag{20}$$

Taking the limit as $y \to \infty$ of the equality

$$H(g_1(y), g_2(y), ..., g_m(y)) = \sum_{i=1}^{m} V_i(g_i(y))$$

yields

$$H(\lim_{y\to\infty} g_1(y), \lim_{y\to\infty} g_2(y), ..., \lim_{y\to\infty} g_m(y)) = \sum_{i=1}^{m} C_i.$$

If H is nonnegative, (20) yields

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{m} |\nabla u_i(x_i)|^2 &= \frac{1}{2}(|a|^2 + 1)\sum_{i=1}^{m} g_i'(y)^2 \\
&\leq H(g_1(y), g_2(y), ..., g_m(y)) \\
&= H(u_1(x), u_2(x), ..., u_m(x)).
\end{aligned}$$

Remark 3.5. It is not clear to us whether the approach above provides any more information about H-monotone solutions when the solutions are not 1-dimensional, but surprising it does imply more about solutions with the *opposite* geometry. Suppose that $-H$ is orientable, and that u is a bounded $-H$ monotone solution to (13); in other words, for all $i \neq j$

$$\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial u_i}{\partial x_N}\frac{\partial u_j}{\partial x_N} > 0.$$

Then for each x', the measure $\gamma^{x'}$ *maximizes* (rather than minimizes) the multi-marginal Kantorovich functional (2) and the functions V_i from Theorem 3.1 satisfy the constraint

$$\sum V_i(x', p_i) \geq H(p_1, p_2, ..., p_m). \tag{21}$$

If the u_i also have common level sets, then the V_i are independent of x', by Theorem 3.1. If H is everywhere nonnegative, taking infimums in (21) yields $\sum_{i=1}^{m} \bar{V}_i \geq 0$; combining this with the previous corollary implies the following Modica estimate for these types of solutions:

$$\frac{1}{2}\sum_{i=1}^{m} |\nabla u_i(x_i)|^2 \leq H(u).$$

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